Graph-Theoretic Analysis of Belief System Dynamics under Logic Constraints[1]

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1. INTRODUCTION

The modeling of opinion dynamics spans several decades of interdisciplinary research [Converse 2006; Feldman 1988; Acemoglu et al. 2011; Jackson 2010; Hegselmann and Krause 2005; Mirtabatabaei and Bullo 2012; Friedkin 2015; Cartwright 1959; Friedkin and Johnsen 2011]. Belief systems are typically modeled as a process where agents continuously update their opinions on a set of truth statements via repeated interactions, and opinions are exchanged following some social structure [DeGroot 1974; Abelson 1964]. New opinions are formed by aggregating operations weighted by the relative importance assigned by an individual to others. This simple characterization has provided tools for analyzing the long-term behavior of belief systems using systems theory. Nevertheless, without significant modification, this framework has been shown insufficient to explain the existence of shared beliefs in a population [Friedkin et al. 2016].

Recently proposed generalizations of opinion dynamics models integrate functional interdependencies among issues that coherently bound ideas and attitudes [Parsegov et al. 2017]. The existence of *logic constraints* in a belief system provides a successful model for the evolution of opinions in both large-scale populations and small groups [Friedkin et al. 2016]. *Logic constraints* build upon the natural idea that believing a specific statement is true may depend on the belief that some other related statements are true as well.

Understanding the role of the networks involved in the structural features of a belief system is of critical importance and can have direct implications for better decision-making and policy design [Butts 2016; Amblard and Deffuant 2004; Fortunato 2005; van der Linden 2017]. Particularly, the fields on consensus dynamics and average consensus theory have provided a plethora of results analyzing the impact of the network structure with the opinion dynamic of a social network [Xia and Cao 2017; Egerstedt et al. 2012; Yu et al. 2010; Proskurnikov and Tempo 2018]. Even though sophisticated algebraic tools [Proskurnikov et al. 2017] exists for the analysis of opinion dynamics, they can be unpractical or intractable for large-scale complex networks.

We study how the structural properties of the social network of agents and the set of logic constraints influence the dynamics of a belief system from a *graph-theoretic* point of view. Informally, we answer the following three questions with graph-theoretic conditions that are easily accessible for a number of commonly used topologies in large-scale complex networks: (1) When does a belief system converge? (2) How long does it take for a belief system to converge? (3) Where does a belief system converge?

2. BELIEF SYSTEMS WITH LOGIC CONSTRAINTS

Friedkin et al. [2016] describe a belief system with logic constraints as a group of n agents that periodically exchange and update their opinions about a set of m different truth statements with logical

[1] Based on Nedić et al. [2019]

dependencies among them. After each social interaction, the agents use shared opinions, as well as underlying logical dependencies among them, to update their beliefs.

The agents exchange their opinions by interacting over a social network captured by a graph $\mathcal{G} = (V, E)$, where V is the set of agents, and E is a set of edges. A directed edge towards an agent indicates that it receives the opinion of another agent, i.e., the directed flow of information. Analogously, the logical dependencies among the truth statements are modeled by a graph $\mathcal{T} = (W, D)$, where an edge between two statements exists if the belief in one statement affects belief in the other.

The generalized dynamics of a belief system are defined as follows. First, every agent aggregates its opinions on every truth statement according to the imposed logic constraints (i.e., modifying the opinions to consider the dependencies on the other truth statements). Second, the agents share their opinions over a social network, where the opinions are aggregated again to take into account those coming from the neighboring agents (i.e., social interactions). Finally, a new opinion is formed as a combination of the most recent aggregation and the initial opinion, modeling the adversity to deviate from the initial beliefs or stubbornness.

The aggregation steps consist of weighted (convex) combinations of the available values, where the weights represent the relative influence. This model is described as follows for an arbitrary agent $i \in V$ and an arbitrary statement $u \in W$:

$$\hat{x}^i_k(u) = \sum_{v=1}^{m} C_{uv} x^i_k(v) \qquad \text{(Aggregation by logic constraints)} \qquad (1a)$$

$$\bar{x}^i_k(u) = \sum_{j=1}^{n} A_{ij} \hat{x}^j_k(u) \qquad \text{(Aggregation by social network)} \qquad (1b)$$

$$x^i_{k+1}(u) = \lambda^i \bar{x}^i_k(u) + (1-\lambda^i) x^i_0(u) \qquad \text{(Influence of initial beliefs)} \qquad (1c)$$

where $0 \leq x^i_k(u) \leq 1$ represents the opinion of an agent i at time k on a certain statement u, while $\hat{x}^i_k(u)$ and $\bar{x}^i_k(u)$ are the intermediate aggregation steps. The opinion of an agent on a specific statement being true or false is modeled by a scalar value between zero and one. A value of zero indicates that the given agent strongly believes a specific statement is false, whereas a value of one indicates that the agent believes the statement is true. Similarly, a value of 0.5 indicates the maximal uncertainty about a statement.

The intermediate aggregated opinion $\hat{x}^i_k(u)$ of agent i on statement u is formed by using the opinions of the same agent about the other statements v. The parameters $0 \leq C_{uv} \leq 1$ are compliant with the graph $\mathcal{T}$ that models the logic constraints in the sense that C_{uv} is nonzero if the statement u depends on statement v, and otherwise $C_{uv} = 0$. These parameters represent the strength of the logic constraints, i.e., the influence that an opinion on a statement has on the opinion on other statements. Subsequently, the intermediate aggregated opinion $\bar{x}^i_k(u)$ of agent i on statement u is formed by combining all the intermediate opinions $\bar{x}^i_k(u)$ of neighboring agents j. In this update, the parameters $0 \leq A_{ij} \leq 1$ represent the weights that an agent i assigns to the information coming from its neighbor j, for example A_{13} is how agent 1 weights the opinions shared by agent 3. These parameters are compliant with the network $\mathcal{G}$ in the sense that if there is an incoming edge to agent i from agent j in the graph, then the corresponding weight A_{ij} is nonzero.

Equation (1c) indicates that, at time $k+1$, the new opinion $x^i_{k+1}(u)$ of agent i on statement u is obtained as a weighted combination of its intermediate aggregated opinion $\bar{x}^i_k(u)$ at time k and its initial opinion $x^i_0(u)$ on statement u. The parameter $0 \leq \lambda^i \leq 1$ that agent i uses models its stubbornness. If $\lambda^i < 1$ we say an agent is *stubborn*, where $\lambda^i = 0$ indicates that the agent i is *maximally closed* to the influence of others. If $\lambda^i = 1$, agent i is said to be *maximally open* to the influence of others, and *oblivious* if additionally, it is not influenced by stubborn agents. Note that when there are stubborn agents, these can be viewed as leaders in the network, given that their opinions directly influence the final belief values in the network.

3. RESULTS

3.1 When does a Belief System Converge?

The convergence of the belief system can be stated as a question of the existence of a limit of the beliefs, as the social interactions continue with time. That is, whether there exists a vector of opinions x_∞ such that

$$\lim_{k \to \infty} x_k = x_\infty,$$

for any initial value x_0.

THEOREM 3.1. *The process* (1) *converges to equilibrium if an only if every closed strongly connected component of the graph $\mathcal{T}$ is aperiodic and every closed strongly connected component of the graph $\mathcal{G}$ composed by oblivious agents only is aperiodic.*

3.2 How long does a Belief System take to Converge?

We seek to characterize the time required by the process in equation (1) to be arbitrarily close to its limiting value in terms of properties of the graphs $\mathcal{G}$ and $\mathcal{T}$, such as the number of agents and truth statements, and the topology of the graphs.

THEOREM 3.2. *Assume the process* (1) *converges to equilibrium. Moreover, let $L_\mathcal{T}$ and $H_\mathcal{T}$ be the maximum expected coupling time and maximum absorbing time of the closed aperiodic and strongly connected components of the graph $\mathcal{T}$, and let $L_\mathcal{G}$ and $H_\mathcal{G}$ be the maximum expected coupling time and absorbing time of a closed aperiodic and strongly connected components of the graph $\mathcal{G}$ composed by oblivious agents only. Then, for $k \geq 32(\max\{L_\mathcal{T}, L_\mathcal{G}\} + \max\{H_\mathcal{T}, H_\mathcal{G}\}) \log(1/\epsilon)$, it holds for the belief system in* (1) *that $\|x_k - x_\infty\|_{TV} \leq \epsilon$.*

3.3 Where Does a Belief System Converge?

So far we have discussed the conditions for convergence of a belief system and the corresponding convergence time. Convergence implies the existence of a vector x_∞ where the set of beliefs settles as the number of interactions increases. We are interested in a characterization of this limit vector that admits a rapid computation of its value.

THEOREM 3.3. *Assume the process* (1) *converges to equilibrium. Let S be a strongly connected component of the system graph $\mathcal{P}$, with factors A_S and C_S, i.e., $P_S = A_S \otimes C_S$. If S is closed, then,*

$$\lim_{k \to \infty} x_k^i = (\pi_{A_S} \otimes \pi_{C_S})'(x_0^{A_S} \otimes x_0^{C_S}) \qquad \forall i \in S.$$

Moreover, if S is open with edges coming from the set of nodes S^M, then

$$\lim_{k \to \infty} x_k^i = \sum_{j \in S^M} p_{ij} x_\infty^j \qquad \forall i \in S,$$

where p_{ij} is the probability of absorption of a random walk starting at node i into a node $j \in S^M$ with limiting value x_∞^j.

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